MEROMORPHIC FUNCTIONS SHARE THREE VALUES WITH THEIR DIFFERENCE OPERATORS

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Abstract. In the work, we focus on a conjecture due to Z.X. Chen and H.X. Yi [1] which is concerning the uniqueness problem of meromorphic functions share three distinct values with their difference operators. We prove that the conjecture is right for meromorphic function of finite order. Meanwhile, a result of J. Zhang and L.W. Liao [10] is generalized from entire functions to meromorphic functions.

1. Introduction and main result

In Nevanlinna theory, to consider the relationship of two meromorphic functions if they share several values CM or IM is an important subtopic, such as the famous Nevanlinna's five and four values theorems [5]. In 1976, L.A. Rubel and C.C. Yang [6] showed that if nonconstant entire function f and its first derivative f' share two distinct values CM, then they are identical. This result is extended by E. Mues and N. Steinmetz [4] in 1979 from shared values CM to IM, by L.Z Yang [7] in 1990 from first derivative to k-th derivative.

As the difference analogues of Nevanlinnas theory are being investigated, J. Zhang and L.W. Liao [10] considered the difference analogues of the result of L.A. Rubel and C.C. Yang. They replaced the derivative f' by the difference operator $\Delta f = f(z+1) - f(z)$ and obtained the following result.

Theorem A. *Let f be a transcendental entire function of finite order and a, b be two distinct constants. If $\Delta f(\not\equiv 0)$ and f share a, b CM, then $\Delta f = f$. Furthermore, f must be of the following form $f(z) = 2^z h(z)$, where h is a periodic entire function with period 1.*

In 2013, under the restriction on the order of meromorphic function, Z.X. Chen and H.X. Yi [1] deduced a uniqueness theorem of meromorphic functions share three distinct values with their difference operators as follows.

Theorem B. *Let f be a transcendental meromorphic function such that its order of growth $\rho(f)$ is not an integer or infinite, let $c \in \mathbb{C}$. If f and $\Delta f(\not\equiv 0)$ share three distinct values $e_1,\ e_2,\ \infty$, then $f(z+c) = 2f(z)$.*

2000 *Mathematics Subject Classification.* 30D35, 39A10.
Key words and phrases. Uniqueness problem, Difference operator, Borel's lemma.
The research was supported by the Natural Science Foundation of Shandong Province Youth Fund Project (ZR2012AQ021).

In the same paper, Z.X. Chen and H.X. Yi conjectured that the restriction on the order of growth of f in Theorem B can be omitted. Clearly, Theorem A showed that the conjecture is right if f is an entire function of finite order. In the present paper, we still focus on the conjecture and prove that it holds if f is a meromorphic function of finite order. In fact, our result is stated as follows.

Main theorem. Let f be a transcendental meromorphic function of finite order, let $\triangle f = f(z+c) - f(z)(\not\equiv 0)$, where c is a finite number. If $\triangle f$ and f share three distinct values $e_1,\ e_2, \infty$ CM, then $f = \triangle f$.

Remark 1. We point out that there exist meromorphic functions satisfying the conditions of Main theorem. For example, $f(z) = e^{zln2}tan(\pi z)$. Obviously, $f = \triangle f = f(z+1) - f(z)$. So, f and $\triangle f$ share $e_1,\ e_2$ and ∞ CM.

Remark 2. The number of shared values cannot be reduced to two. For example, $f(z) = e^{\pi i z}$ and $\triangle f = f(z+1) - f(z) = -e^{\pi i z}$ share $0,\ \infty$ CM. But $f \neq \triangle f$. The example can be seen in [10].

Remark 3. Obviously, our main theorem is an improvement of Theorem A. We also remark that our proof is based on Borel's lemma [2]. We assume that the reader is familiar with the standard notations in the Nevanlinna theory, see ([8, 9]).

2. Some lemmas

To prove our result, we recall the difference analogue of the second main theorem in the value distribution theory.

Lemma 2.1. [3, Theorem 2.4] *Let $c \in \mathbb{C}$, let f be a meromorphic function of finite order with $\Delta f \neq 0$. Let $q \geq 2$, and let $a_1,\ \cdots,\ a_q \in S(f)$ be distinct periodic functions with period c. Then*

$$m(r, f) + \sum_{i=1}^{q} m(r, \frac{1}{f - a_i}) \leq 2T(r, f) - N_{pair}(r, f) + S(r, f),$$

where $N_{pair}(r, f) = 2N(r, f) - N(r, \Delta f) + N(r\frac{1}{\Delta f})$, and the exceptional set associated with $S(r, f)$ is of finite logarithmic measure.

A version of Borel's lemma is also needed.

Lemma 2.2. [2, p. 69-70] *Suppose that $n \geq 2$ and let $f_1,\ f_2, \cdots, f_n$ be meromorphic functions and $g_1,\ g_2, \cdots, g_n$ be entire functions such that*

(1) $\sum_{j=1}^{n} f_j e^{g_j} = 0$,

(2) when $1 \leq j < k \leq n$, $g_j - g_k \neq 0$,

(3) $T(r, f_j) = o(T(r, \exp\{g_h - g_k\}))$ $(r \to \infty,\ r \notin E)$, $E \subset [1,\ +\infty)$ of finite logarithmic measure. Then $f_j = 0$ for all $j \in \{1,\ 2,\ \cdots,\ n\}$.

3. Proof of Main theorem

Note that $f,\ \triangle f$ share $e_1,\ e_2, \infty$ CM and f is of finite order. Then, there exist two polynomials $\alpha,\ \beta$ such that

$$\frac{f - e_1}{\triangle f - e_1} = e^{\alpha},\ \frac{f - e_2}{\triangle f - e_2} = e^{\beta}. \tag{3.1}$$

If $e^{\alpha}=1$ or $e^{\beta}=1$, then $f=\triangle f$. If $e^{\alpha}=e^{\beta}$, then

$$\frac{f-e_1}{\triangle f-e_1}=\frac{f-e_2}{\triangle f-e_2},$$

which implies that $f=\triangle f$.

On the contrary, suppose that $f\neq\triangle f$. Then

$$e^{\alpha}\neq 1,\ e^{\beta}\neq 1,\ e^{\alpha}\neq e^{\beta}. \tag{3.2}$$

Our aim below is to derive a contradiction.

We derive the following expressions from (3.1):

$$f=e_1+(e_2-e_1)\frac{e^{\beta}-1}{e^{\gamma}-1},\ \triangle f=e_2+(e_2-e_1)\frac{1-e^{-\alpha}}{e^{\gamma}-1}, \tag{3.3}$$

where $\gamma=\beta-\alpha$.

Obviously,

$$T(r,f)\leq T(r,e^{\beta})+T(r,e^{\gamma})+S(r,f). \tag{3.4}$$

By the form of $\triangle f$, we have

$$\begin{aligned}\triangle f=e_2+(e_2-e_1)\frac{1-e^{\gamma-\beta}}{e^{\gamma}-1}&=(e_2-e_1)\Big(\frac{e^{\beta(z+c)}-1}{e^{\gamma(z+c)}-1}-\frac{e^{\beta}-1}{e^{\gamma}-1}\Big)\\&=(e_2-e_1)\Big(\frac{\beta_1e^{\beta}-1}{\gamma_1e^{\gamma}-1}-\frac{e^{\beta}-1}{e^{\gamma}-1}\Big),\end{aligned} \tag{3.5}$$

where $\beta_1(z)=e^{\beta(z+c)-\beta(z)}$ and $\gamma_1(z)=e^{\gamma(z+c)-\gamma(z)}$ are small functions of e^{β} and e^{γ}, respectively.

We claim that $\deg\beta=\deg\gamma$.

If $\deg\beta<\deg\gamma$, then e^{β} is a small function of e^{γ}. Suppose that z_0 is a zero of $\gamma_1e^{\gamma}-1$, not a zero of $\beta_1e^{\beta}-1$. Then, it follows from (3.5) that z_0 is also a zero of $e^{\gamma}-1$. Then z_0 is a zero of γ_1-1. If $\gamma_1-1\neq 0$, then

$$\begin{aligned}T(r,e^{\gamma})&=\overline{N}(r,\frac{1}{\gamma_1e^{\gamma(z)}-1})+S(r,e^{\gamma})\\&\leq N(r,\frac{1}{\beta_1e^{\beta(z)}-1})+N(r,\frac{1}{\gamma_1-1})+S(r,e^{\gamma})=S(r,e^{\gamma}),\end{aligned}$$

a contradiction. Thus, $\gamma_1(z)=e^{\gamma(z+c)-\gamma(z)}=1$. It means that $\deg\gamma=1$. Note that $\deg\beta<\deg\gamma$, so β is a constant, and say A. Thus, again by (3.5), we derive that

$$\triangle f=(e_2-e_1)\Big(\frac{\beta_1e^{\beta}-1}{\gamma_1e^{\gamma}-1}-\frac{e^{\beta}-1}{e^{\gamma}-1}\Big)=(e_2-e_1)\frac{A-A}{e^{\gamma}-1}=0,$$

a contradiction.

If $\deg\beta>\deg\gamma$, then e^{γ} is a small function of e^{β}. Assume that z_0 is zero of $e^{\beta}-1$, not a zero of $e^{\gamma}-1$. Then, z_0 is a zero of $f-e_1$. Note that f and $\triangle f$ share e_1 CM, so z_0 is also a zero of $\triangle f-e_1$. Put z_0 into last form of $\triangle f$ in (3.5), we have

$$e_1=(e_2-e_1)\frac{\beta_1-1}{\gamma_1e^{\gamma}-1}\Big|_{z_0}.$$

Obviously, $e_1 = (e_2 - e_1)\frac{\beta_1 - 1}{\gamma_1 e^{\gamma} - 1}$. Otherwise,

$$\begin{aligned} T(r, e^{\beta}) &= \overline{N}(r, \frac{1}{e^{\beta} - 1}) + S(r, e^{\beta}) \\ &\leq N(r, \frac{1}{e^{\gamma} - 1}) + N(r, \frac{1}{(e_2 - e_1)\frac{\beta_1 - 1}{\gamma_1 e^{\gamma} - 1} - e_1}) + S(r, e^{\beta}) = S(r, e^{\beta}), \end{aligned}$$

a contradiction. Thus,

$$e_1 = (e_2 - e_1)\frac{\beta_1 - 1}{\gamma_1 e^{\gamma} - 1}.$$

Rewrite it as

$$(e_2 - e_1)e^{\beta(z+c) - \beta(z)} - (e_2 - e_1) = e_1 e^{\gamma(z+c)} - e_1. \tag{3.6}$$

We will prove that γ is a constant. On the contrary, suppose that $\deg \gamma \geq 1$. Then, combining (3.6) and $\deg \beta > \deg \gamma$, we obtain that

$$(e_2 - e_1)\beta_1 = (e_2 - e_1)e^{\beta(z+c) - \beta(z)} = e_1 e^{\gamma(z+c)}, \ e_2 - e_1 = e_1. \tag{3.7}$$

It implies $\beta_1 = e^{\gamma(z+c)}$. Rewrite (3.5) as

$$\begin{aligned} &e_2 e^{\beta}(\gamma_1 e^{\gamma} - 1)(e^{\gamma} - 1) + (e_2 - e_1)(e^{\beta} - e^{\gamma})(\gamma_1 e^{\gamma} - 1) \\ =&(e_2 - e_1)[(\beta_1 e^{\beta} - 1)e^{\beta}(e^{\gamma} - 1) - (e^{\beta} - 1)e^{\beta}(\gamma_1 e^{\gamma} - 1)]. \end{aligned}$$

Rewrite it as

$$a_0 e^{2\beta} + a_1 e^{\beta} + a_2 = 0,$$

where $a_0 = (e_2 - e_1)[\beta_1(e^{\gamma} - 1) - (\gamma_1 e^{\gamma} - 1)]$, a_1, a_2 are small functions of e^{β}. It indicates that $a_0 = 0$. Thus,

$$\beta_1(e^{\gamma} - 1) = \gamma_1 e^{\gamma} - 1. \tag{3.8}$$

Put $\beta_1 = e^{\gamma(z+c)}$ into (3.8), we have

$$e^{\gamma(z+c) + \gamma(z)} - 2e^{\gamma(z+c)} + 1 = 0,$$

which implies that γ is a constant, a contradiction. Thus, we obtain that γ is a constant. The form of f shows that f is an entire function. Then, it follows from Theorem A that $f = \triangle f$, a contradiction.

Thus, we prove that

$$\deg \beta = \deg \gamma \geq 1.$$

Still set $e^{\beta(z+c)} = \beta_1 e^{\beta}$ and $e^{\gamma(z+c)} = \gamma_1 e^{\gamma}$, where $\beta_1, \ \gamma_1$ are two small functions of e^{β} and e^{γ}. Then, due to the forms of $f, \ \triangle f$, a routine calculation leads to

$$b_0 e^{2\gamma} + b_1 e^{\beta + 2\gamma} + b_2 e^{\beta + \gamma} + b_3 e^{2\beta} + b_4 e^{2\beta + \gamma} + b_5 e^{\beta} + b_6 e^{\gamma} = 0,$$

where

$$\begin{cases} b_0(z) = (e_2 - e_1)\gamma_1, \ b_1(z) = -e_2\gamma_1, \\ b_2(z) = -(e_2 - e_1) + e_2(\gamma_1 + 1) \\ b_3(z) = (e_2 - e_1)(1 - \beta_1), b_4(z) = (e_2 - e_2)(\beta_1 - \gamma_1), \\ b_5(z) = -e_2, \ b_6(z) = -(e_2 - e_1). \end{cases}$$

Obviously, b_i $(0 \leq i \leq 6)$ are small functions of e^{β} and e^{γ}. (In fact, for the proof of this result, we just need the specific forms of b_0 or b_6.) Rewrite it as

$$\sum_{i=0}^{6} b_i e^{g_i} = 0,$$

where

$$\begin{cases} g_0(z) = 2\gamma, \\ g_1(z) = \beta + 2\gamma, \ g_2(z) = \beta + \gamma, \\ g_3(z) = 2\beta, \ g_4(z) = 2\beta + \gamma, \\ g_5(z) = \beta, \ g_6(z) = \gamma. \end{cases}$$

Suppose that

$$\deg(\beta) = \deg(\gamma) = n.$$

We claim for any $0 \le j < i \le 6$

$$\deg(g_i - g_j) = n.$$

In the following, we consider several cases to prove the above claim.

Case 1. $i = 6$.

It is easy to check that

$$\deg(g_6 - g_4) = \deg(-2\beta) = n,$$
$$\deg(g_6 - g_2) = \deg(-\beta) = n, \ \deg(g_6 - g_0) = \deg(\gamma) = n.$$

Suppose that $\deg(g_6 - g_5) = \deg(\gamma - \beta) < n$. Then $e^{\gamma-\beta}$ is a small function of e^{β} and e^{γ}. We denote by $N_E(r)$ the counting function of those common zeros of $e^{\beta} - 1$ and $e^{\gamma} - 1$. We firstly prove that $N_E(r) = S(r, e^{\gamma})$. Otherwise, suppose that $N_E(r) \neq S(r, e^{\gamma})$. Assume that z_0 is a common zero of $e^{\beta} - 1$ and $e^{\gamma} - 1$. Then z_0 is a zero of $e^{\gamma-\beta} - 1$. If $e^{\gamma-\beta} - 1 \neq 0$, then

$$N_E(r) \le N(r, \frac{1}{e^{\gamma-\beta} - 1}) = S(r, e^{\gamma}),$$

a contradiction. Thus, $e^{\gamma-\beta} - 1 = 0$. It means that $e^{\beta} = e^{\gamma}$. So, the form of f yields that f is a constant, which is impossible. Thus, we prove that $N_E(r) = S(r, e^{\gamma})$.

Without loss of generality, assume that z_0 is a zero of $\gamma_1 e^{\gamma} - 1$, not a zero of $\beta_1 e^{\beta} - 1$. It follows from (3.5) that z_0 is also a zero of $e^{\gamma} - 1$. Then z_0 is a zero of $\gamma_1 - 1$. If $\gamma_1 - 1 \neq 0$, then

$$\begin{aligned} T(r, e^{\gamma}) &= \overline{N}(r, \frac{1}{\gamma_1 e^{\gamma} - 1}) + S(r, e^{\gamma}) \\ &\le N_E(r) + N(r, \frac{1}{\gamma_1 - 1}) + S(r, e^{\gamma}) \\ &\le T(r, \gamma_1 - 1) + S(r, e^{\gamma}) = S(r, e^{\gamma}), \end{aligned}$$

a contradiction. Thus, $\gamma_1 = e^{\gamma(z+c)-\gamma(z)} = 1$, which implies that $e^{\gamma(z+c)} = e^{\gamma(z)}$ and $\deg \gamma = 1$. Then $\deg(\beta - \gamma) < 1$. It means that $\beta - \gamma$ is a constant, say A. Thus, it follows from $e^{\gamma(z+c)} = e^{\gamma(z)}$ that

$$e^{\beta(z+c)-\beta(z)} = e^{\beta(z+c)-\gamma(z+c)-(\beta(z)-\gamma(z))} = e^{A-A} = 1,$$

So $e^{\beta(z+c)} = e^{\beta(z)}$. Then, by (3.5) we get $\triangle f = 0$, a contradiction. Thus,

$$\deg(g_6 - g_5) = n.$$

Suppose that $\deg(g_6 - g_3) = \deg(\gamma - 2\beta) < n$. The notation $N_E(r)$ is defined as above discussion. We firstly prove that $N_E(r) = S(r, e^{\gamma})$. Otherwise, suppose that $N_E(r) \neq S(r, e^{\gamma})$. Without loss of generality, assume that z_0 is a common zero of

$e^{\beta}-1$ and $e^{\gamma}-1$. Then $e^{\gamma(z_0)}=1$ and $e^{\beta(z_0)}=1$. Furthermore, $e^{\gamma(z_0)-2\beta(z_0)}=1$. Clearly, $e^{\gamma-2\beta}$ is a small function of e^{γ}. If $e^{\gamma-2\beta}-1\neq 0$, then

$$N_E(r)\leq N(r,\frac{1}{e^{\gamma-2\beta}-1})=S(r,e^{\gamma}),$$

a contradiction. Thus, $e^{\gamma-2\beta}=1$ and $e^{\gamma}=e^{2\beta}$. Then,

$$\begin{aligned}\triangle f&=e_2+(e_2-e_1)\frac{1-e^{\gamma-\beta}}{e^{\gamma}-1}=e_2+(e_2-e_1)\frac{e^{\beta}(e^{-\beta}-e^{\gamma-2\beta})}{e^{\gamma}-1}\\&=e_2+(e_2-e_1)\frac{1-e^{\beta}}{e^{\gamma}-1}.\end{aligned}$$

From the forms of f and Δf, we have

$$f-e_1=-(\triangle f-e_2). \tag{3.9}$$

Since f and $\triangle f$ share e_1 and e_2 CM, it follows from (3.9) that $e_1,\ e_2$ are two Picard values of f. Then, by the second main theorem (see Lemma 2.1), we get

$$\begin{aligned}T(r,f)&\leq N(r,f)+N(r,\frac{1}{f-e_1})+N(r,\frac{1}{f-e_2})\\&-2N(r,f)+N(r,\Delta f)-N(r,\frac{1}{\Delta f})+S(r,f)\\&\leq N(r,\frac{1}{f-e_1})+N(r,\frac{1}{f-e_2})+S(r,f)\leq S(r,f),\end{aligned}$$

a contradiction. Thus, $N_E(r)=S(r,e^{\gamma})$.

Similarly as above, we can deduce that $\gamma_1=1$ and $\deg\gamma=1$. Then, by $\deg(\gamma-2\beta)<n$ and $\deg\gamma=\deg\beta$, we can set $e^{\beta}=AH$, $e^{\gamma}=BH^2$ and $e^{\gamma-\beta}=CH$, where $A,\ B,\ C$ are three nonzero constants. From (3.5), a careful calculation leads to

$$e_2e^{\gamma}-(e_2-e_1)e^{\gamma-\beta}-e_1=(e_2-e_1)(\beta_1-1)e^{\beta}.$$

Rewrite the above equation as

$$e_2BH^2+b_1H-e_1=0,$$

where $b_1=-(e_2-e_1)[C+A(\beta_1-1)]$ is a small function of H. Then $e_2=0$ and $e_1=0$. It is impossible. Thus,

$$\deg(g_6-g_3)=\deg(\gamma-2\beta)=n.$$

Suppose that $\deg(g_6-g_1)=\deg[-(\gamma+\beta)]<n$. The notation $N_E(r)$ is defined as above discussion. We firstly prove that $N_E(r)=S(r,e^{\gamma})$. Otherwise, suppose that $N_E(r)\neq S(r,e^{\gamma})$. Without loss of generality, assume that z_0 is a common zero of $e^{\beta}-1$ and $e^{\gamma}-1$. Then $e^{\gamma(z_0)}=1$ and $e^{\beta(z_0)}=1$. Furthermore, $e^{\gamma(z_0)+\beta(z_0)}=1$. Clearly, $e^{\gamma+\beta}$ is a small function of e^{γ}. If $e^{\gamma+\beta}-1\neq 0$, then

$$N_E(r)\leq N(r,\frac{1}{e^{\gamma+\beta}-1})=S(r,e^{\gamma}),$$

a contradiction. Thus, $e^{\gamma+\beta}=1$ and $e^{-\gamma}=e^{\beta}$. Then,

$$\begin{aligned}f&=e_1+(e_2-e_1)\frac{e^{\beta}-1}{e^{\gamma}-1}=e_1+(e_2-e_1)\frac{e^{-\gamma}-1}{e^{\gamma}-1}\\&=e_1+(e_1-e_2)e^{-\gamma},\end{aligned}$$

and

$$\triangle f = e_2 + (e_2 - e_1)\frac{1 - e^{\gamma-\beta}}{e^\gamma - 1} = e_2 + (e_2 - e_1)\frac{1 - e^{2\gamma}}{e^\gamma - 1}$$
$$= e_1 + (e_1 - e_2)e^\gamma.$$

Furthermore, by $\triangle f = f(z + c) - f(z)$, we have

$$e_1 + (e_1 - e_2)e^\gamma = (e_2 - e_1)(e^{-\gamma(z+c)} - e^{-\gamma}).$$

Rewrite it as

$$e_1 + (e_1 - e_2)e^\gamma = (e_2 - e_1)(\gamma_2 e^{-\gamma} - e^{-\gamma}),$$

where γ_2 is a small function of e^γ and $e^{-\gamma}$. Then, it implies that $e_1 - e_2 = 0$, a contradiction. Thus, $\deg(g_6 - g_1) = n$.

Case 2. $i = 5$.

It is obvious from the above discussion that

$$\deg(g_5 - g_4) = \deg(\beta + \gamma) = n,\ \deg(g_5 - g_3) = \deg(-\beta) = n,$$
$$\deg(g_5 - g_2) = \deg(-\gamma) = n,\ \deg(g_5 - g_1) = \deg(-2\gamma) = n.$$

Suppose that $\deg(g_5 - g_0) = \deg(\beta - 2\gamma) < n$. The notation $N_E(r)$ is defined as above discussion. We firstly prove that $N_E(r) = S(r, e^\gamma)$. Otherwise, suppose that $N_E(r) \neq S(r, e^\gamma)$. Without loss of generality, assume that z_0 is a common zero of $e^\beta - 1$ and $e^\gamma - 1$. Then $e^{\gamma(z_0)} = 1$ and $e^{\beta(z_0)} = 1$. Furthermore, $e^{\beta(z_0)-2\gamma(z_0)} = 1$. Clearly, $e^{\beta-2\gamma}$ is a small function of e^γ. If $e^{\beta-2\gamma} - 1 \neq 0$, then

$$N_E(r) \leq N(r, \frac{1}{e^{\beta-2\gamma} - 1}) = S(r, e^\gamma),$$

a contradiction. Thus, $e^{\beta-2\gamma} = 1$ and $e^\beta = e^{2\gamma}$. Then,

$$f = e_1 + (e_2 - e_1)\frac{e^\beta - 1}{e^\gamma - 1} = e_1 + (e_2 - e_1)\frac{e^{2\gamma} - 1}{e^\gamma - 1}$$
$$= e_2 + (e_2 - e_1)e^\gamma,$$

and

$$\triangle f = e_2 + (e_2 - e_1)\frac{1 - e^{\gamma-\beta}}{e^\gamma - 1} = e_2 + (e_2 - e_1)\frac{1 - e^{-\gamma}}{e^\gamma - 1}$$
$$= e_1 + (e_1 - e_2)e^{-\gamma}.$$

Furthermore, by $\triangle f = f(z + c) - f(z)$, we have

$$e_1 + (e_1 - e_2)e^{-\gamma} = (e_2 - e_1)(e^{\gamma(z+c)} - e^\gamma),$$

which implies that $e_1 - e_2 = 0$, a contradiction. Thus,

$$\deg(g_5 - g_0) = \deg(\beta - 2\gamma) = n.$$

Case 3. $i = 4$.

It is obvious from the above discussion that

$$\deg(g_4 - g_3) = \deg(\gamma) = n,\ \deg(g_4 - g_2) = \deg(\beta) = n,$$
$$\deg(g_4 - g_1) = \deg(\beta - \gamma) = n,\ \deg(g_4 - g_0) = \deg(2\beta - \gamma) = n.$$

Case 4. $i = 3$.

It is obvious from the above discussion that

$$\deg(g_3 - g_2) = \deg(\beta - \gamma) = n,$$

$$\deg(g_3 - g_1) = \deg(\beta - 2\gamma) = n, \ \deg(g_3 - g_0) = \deg 2(\beta - \gamma) = n.$$

Case 5. $i = 2$.

It is obvious from the above discussion that

$$\deg(g_2 - g_1) = \deg(\gamma) = n, \ \deg(g_2 - g_0) = \deg(\beta - \gamma) = n.$$

Case 6. Obviously, $\deg(g_1 - g_0) = \deg(\beta) = n$.

Thus, the claim is proved. Then, by a Borel' lemma (see Lemma 2.2), we get $b_j = 0$ for $0 \le j \le 6$. But $b_6 = -(e_2 - e_1) \neq 0$, a contradiction.

Thus, the proof of this theorem is finished.

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